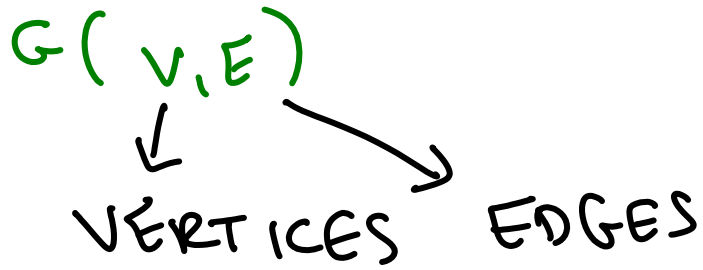


INDUCED SUBGRAPHS AND χ

MARIA CHUDNOVSKY
(PRINCETON)

①



$\chi(G)$ CHROMATIC NUMBER

$\omega(G)$ CLIQUE NUMBER

$$\forall G \quad \chi(G) \geq \omega(G)$$

WHAT ABOUT $\chi(G) \leq f(\omega(G))$?

(2)

WHAT ABOUT $\chi(G) \leq f(\omega(G))$?

FALSE

THM (MYCIELSKI)

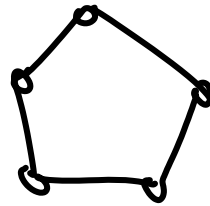
$\forall k \exists G$ s.t. $\omega(G) = 2$ & $\chi(G) \geq k$

MYCIELSKI GRAPHS:

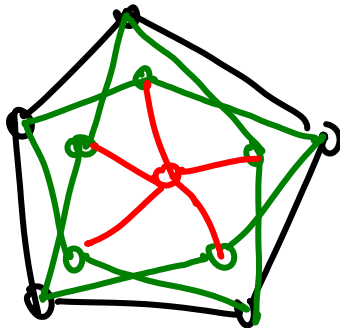
M_2



M_3



M_4



$$\omega(M_k) = 2$$

$$\chi(M_k) = k$$

3

STRONGER THEOREM (ERDÖS)

FOR EVERY g, k THERE IS
A GRAPH WITH NO CYCLE OF
LENGTH $< g$, AND WITH $\chi > k$.

4

WHEN $\exists f$ s.t. $\chi(G) \leq f(\omega(G))$

MORE PRECISELY:

FOR WHICH INFINITE FAMILIES \mathcal{F} OF GRAPHS, DOES THERE EXIST $f_{\mathcal{F}}$ s.t.

$$\chi(G) \leq f_{\mathcal{F}}(\omega(G)) \quad \forall G \in \mathcal{F}$$

5

QUESTION 1:

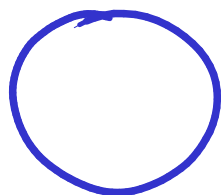
WHEN IS $\chi(G) = \omega(G)$?

BIPARTITE GRAPHS

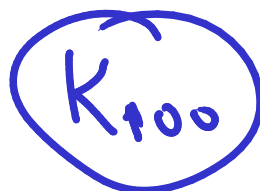
COMPLEMENTS OF BIP

COMPARABILITY GRAPHS

⋮



≤ 99 VERT



(6)

G IS PERFECT IF

$\chi(H) = \omega(H)$ FOR EVERY

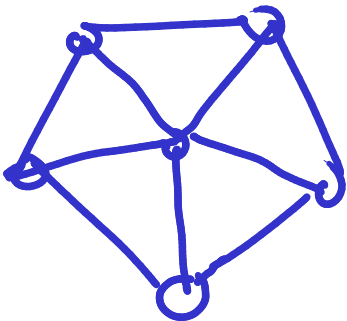
INDUCED SUBGRAPH H OF G

(BERGE, 1961)

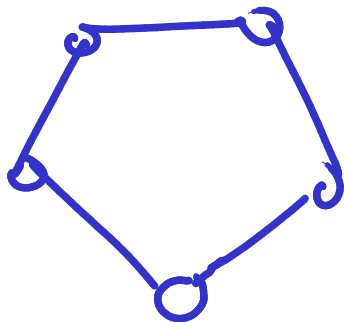
H IS AN INDUCED SUBGRAPH
OF G IF

$V(H) \subseteq V(G)$, AND

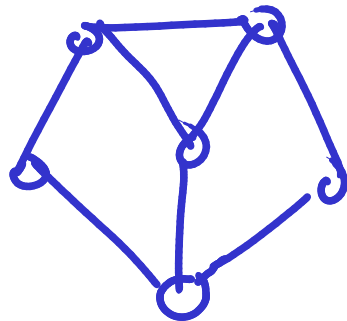
$uv \in E(H)$ IFF $uv \in E(G)$
& $u, v \in V(H)$



G



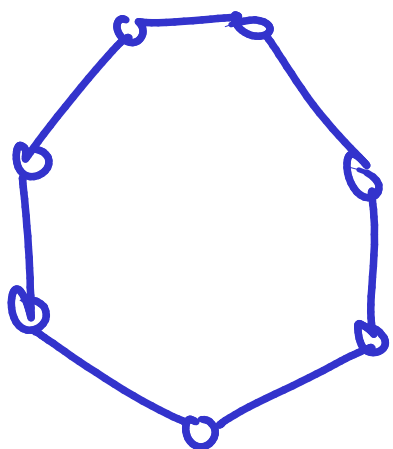
INDUCED
S.G. OF G



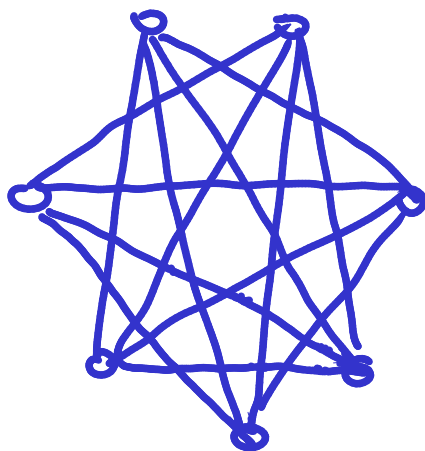
NOT AN
INDUCED S.G.
OF G

7

WHICH GRAPHS ARE NOT PERFECT?



C_{2n+1}
 $n \geq 2$



C_{2n+1}
 $n \geq 2$

8

THE STRONG PERFECT GRAPH THEOREM

(C. ROBERTSON, SEYMOUR, THOMAS)

G IS PERFECT IFF G HAS NO
INDUCED SUBGRAPH ISOMORPHIC TO
 C_{2n+1} OR C_{2n+1}^c WITH $n \geq 2$

9

G IS **BERGE** IF NO INDUCED
SUBGRAPH OF G IS ISOMORPHIC
TO C_{2n+1} OR C_{2n+1}^c FOR $n \geq 2$

SPGT : **BERGE** = **PERFECT**

PROOF (IDEA) : DESCRIBE ALL
BERGE **GRAPHS**

"THM 1" EVERY BERGE GRAPH IS
EITHER BASIC, OR ADMITS A
"USEFUL" DECOMPOSITION

"THM 2" ALL BASIC GRAPHS
ARE PERFECT

"THM 3" LET G BE BERGE,
NOT PERFECT, WITH $|V(G)|$ MIN.
THEN G DOES NOT ADMIT ANY
OF THE DECOMPOSITIONS FROM
"THM 1"

(11)

DEF LET \mathcal{C} BE A CLASS OF GRAPHS CLOSED UNDER TAKING INDUCED SUBGRAPHS.

\mathcal{C} IS χ -BOUNDED IF

$\exists f$ S.T. $\chi(G) \leq f(\omega(G)) \forall G \in \mathcal{C}$

f IS A χ -BOUNDING FUNC FOR \mathcal{C}

FIX H .

$$\text{FORB}(H) = \{G : H \not\subseteq G\}$$

(12)

IS $\text{FORB}(H)$ X -BOUNDED FOR
ALL H ?

NO

THM IF $\text{FORB}(H)$ IS X -BOUNDED,
THEN H HAS NO CYCLES


PROOF:

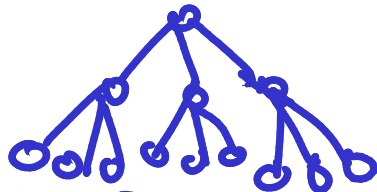
RECALL: $\forall g, k \exists G$ WITH
NO CYCLES OF LENGTH $< g$,
AND WITH $\chi(G) > k$

CONT (GYARFAS; SUMNER)

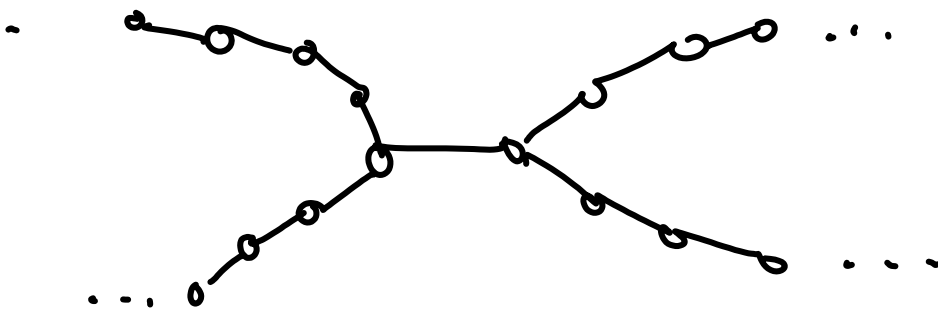
IF H HAS NO CYCLES, THEN
 $FORB(H)$ IS X -BOUNDED

KNOWN CASES:

- H IS A PATH 
(GYARFAS)

- H HAS RADIUS ≤ 2 
(KIERSTEAD & PENRICE)

- TOPOLOGICAL VERSION (SCOTT)



(C., SCOTT, SEYMOUR, SPIKILL)

(14)

A **HOLE** IN G IS AN INDUCED
CYCLE OF LENGTH ≥ 4

QUESTION: FORBID HOLES OF SOME
LENGTHS. DO WE GET A
 χ -BOUNDED FAMILY?

OBSERVATION: NEED TO FORBID
INFINITELY MANY LENGTHS

(15)

THM (ADPARIU-BERRY, C, HAVET,
REED, SEYMOUR)

IF G HAS NO EVEN HOLES,
THEN $\chi(G) \leq 2 \omega(G)$.

(CONJECTURED BY REED)

THM (BONAMY, CHARBIT, THOMASSE)

IF G HAS NO HOLES OF LENGTH
 $0 \pmod 3$ AND NO Δ , THEN

$$\chi(G) \leq 10^6.$$

(CONJECTURED BY KALAI & MESHULAM)

MORE CONJECTURES BY GYÁRFÁS:

CONJ1 (GYÁRFÁS) THE FAMILY OF GRAPHS WITH NO ODD HOLE IS χ -BOUNDED

CONJ2 (GYÁRFÁS)

$\forall \ell$ THE FAMILY OF GRAPHS WITH NO HOLE OF LENGTH $\geq \ell$ IS χ -BOUNDED

CONJ3 (GYÁRFÁS)

$\forall \ell$ THE FAMILY OF GRAPHS WITH NO ODD HOLE OF LENGTH $\geq \ell$ IS χ -BOUNDED

THM 1 (SCOTT & SEYMOUR, 2014)

THE FAMILY OF ODD-HOLE-FREE
GRAPHS IS χ -BOUNDED

(CONT'D BY GYÁRFÁS 1987)

CONJ (HOANG MCDIARMID)

LET G BE AN ODD-HOLE-FREE GRAPH.

THEN $V(G) = V_1 \cup V_2$ S.T.

$\forall i = 1, 2 \quad \omega(G|V_i) \leq \omega(G) - 1$

(AND THEREFORE $\chi(G) \leq 2^{\omega(G)}$)

THM (C, SIVARAMAN)

A VARIANT IS TRUE IF G

IS BULLFREE

THM 2 (C., SCOTT, SEYMOUR, 2015) 18

FOR EVERY ℓ THE FAMILY OF
GRAPHS WITH NO HOLE OF
LENGTH $\geq \ell$ IS χ -BOUNDED

THM 3 (C., SCOTT, SEYMOUR, SPIRICK,
2016)

FOR EVERY ℓ THE FAMILY OF
GRAPHS WITH NO ODD HOLE OF
LENGTH $\geq \ell$ IS χ -BOUNDED

19

THM (GYÁRFA'S)

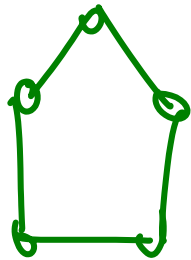
$\forall \ell$ THE FAMILY OF GRAPHS WITH
NO INDUCED PATH OF LENGTH $\geq \ell$
IS χ BOUNDED

PF GROW A PATH INTO A
COMPONENT WITH BIG χ

(GYÁRFA'S PATH)

THM (SCOTT, ~1995)

$\forall \ell$ THE FAMILY OF GRAPHS WITH
NO C_5 AND NO HOLE OF
LENGTH $\geq \ell$ IS χ -BOUNDED



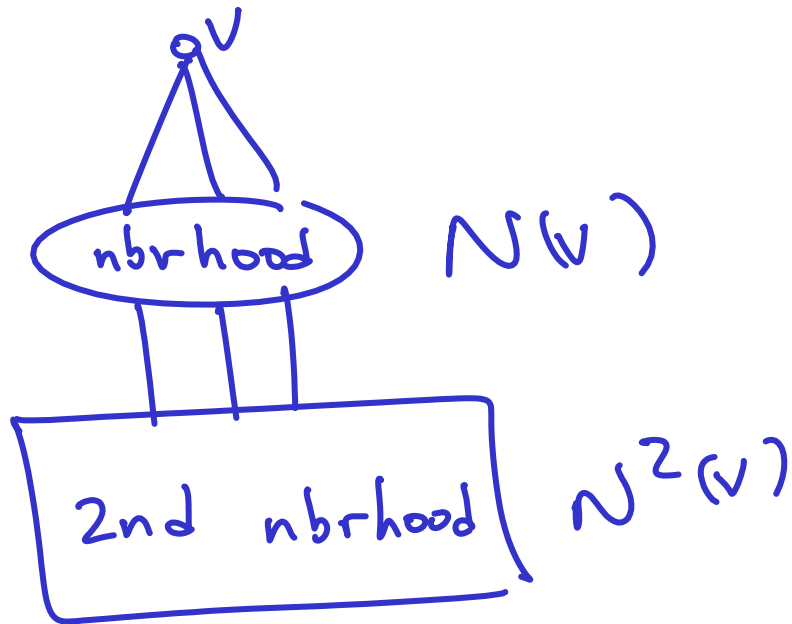
C_5

∴

PROOF

INDUCTION ON $\omega(G)$

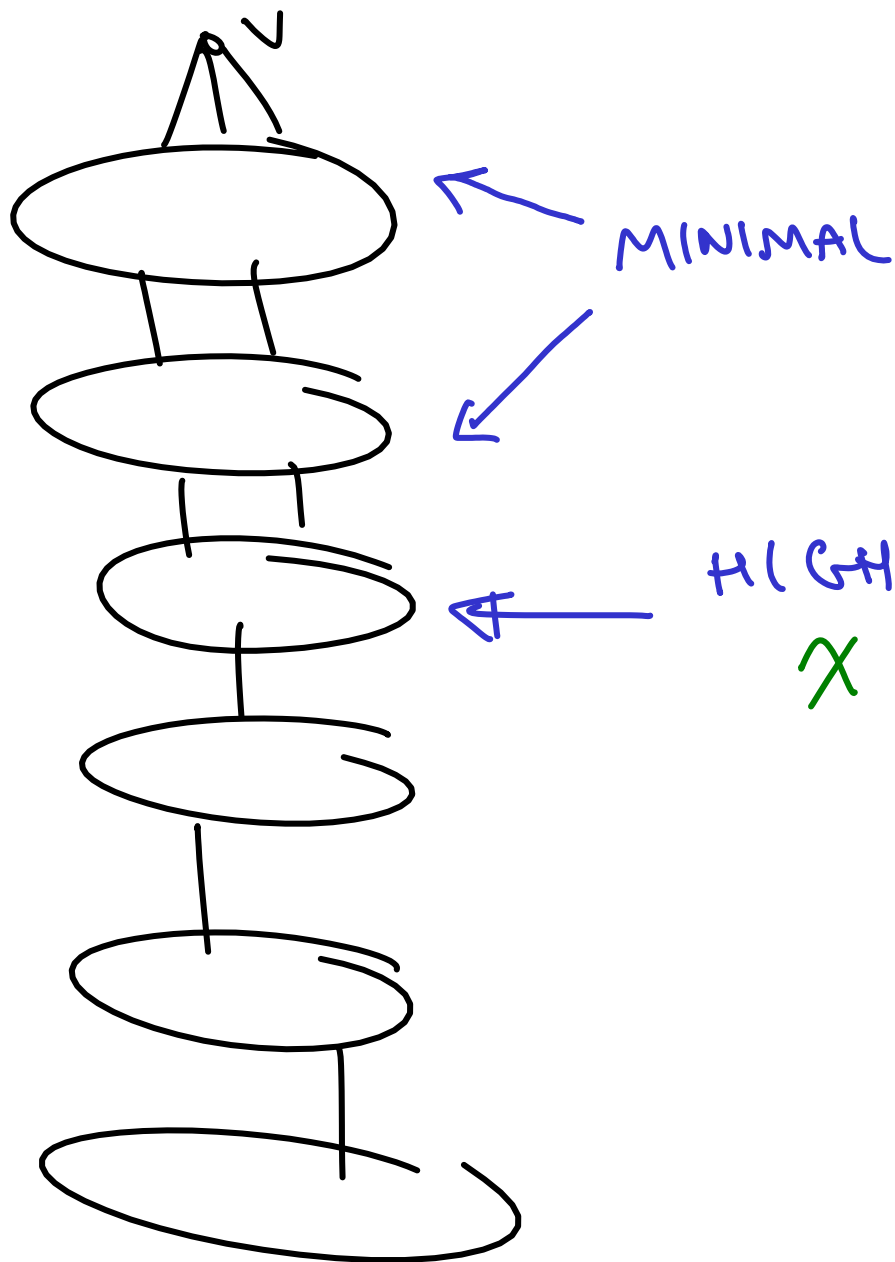
(2)



IDEA : LAYERING

(22)

ASSUME: $\chi(G)$ BIG

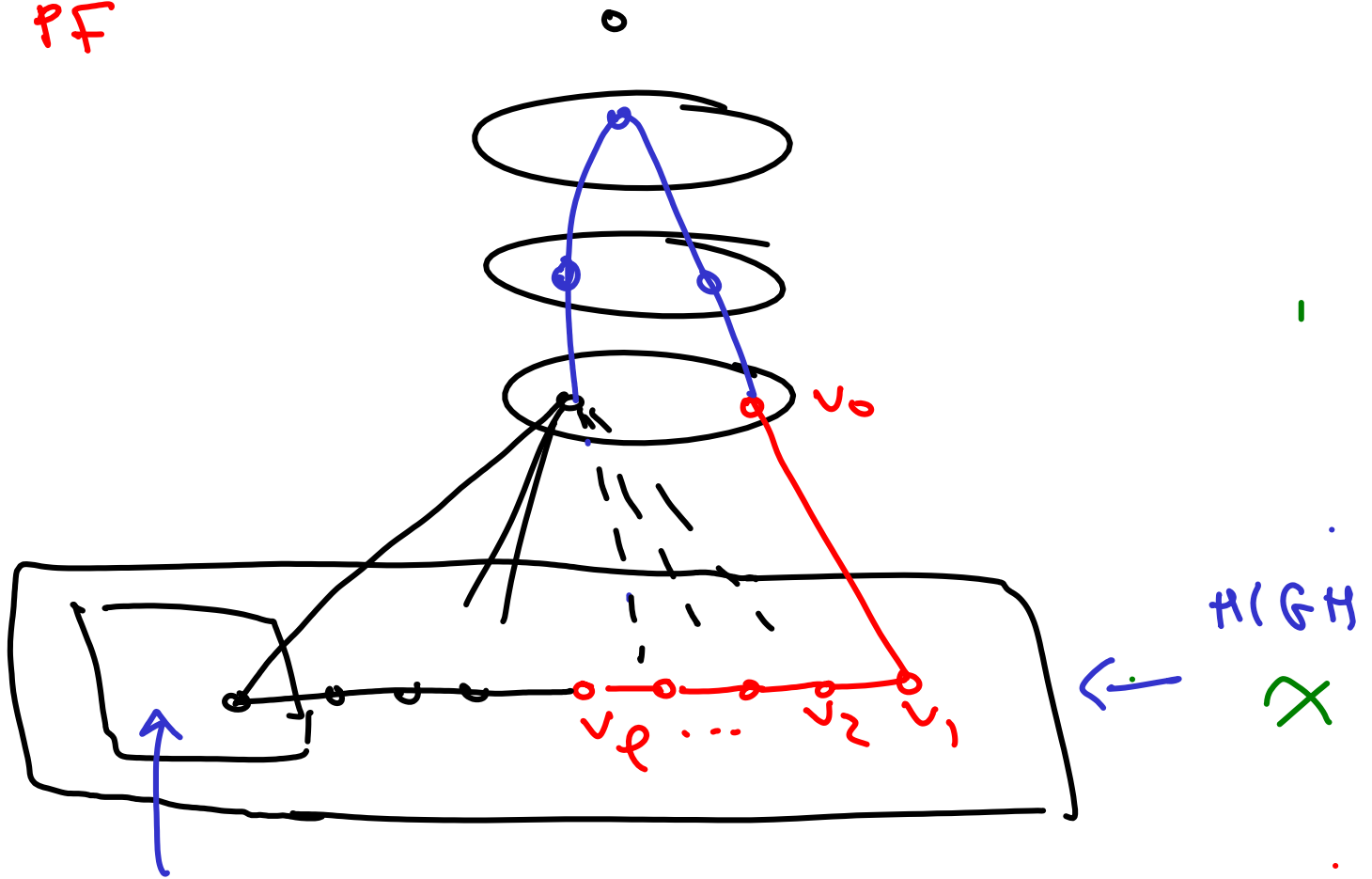


STEP 1

(23)

THM (SCOTT) LET $k, \ell \in \mathbb{Z}^+$, LET \mathcal{F} BE A FAMILY OF GRAPHS WITH NO HOLE OF LENGTH $\geq \ell$, AND S.T. $\chi(N^2(v)) \leq k \quad \forall v \in V(G) \quad \forall G \in \mathcal{F}$.
 THEN $\chi(G) \leq f(k, \ell, \omega(G)) \quad \forall G \in \mathcal{F}$.

PF



HIGH χ
 S.G. OF $G \setminus \bigcup_{i=0}^{\ell} N^2(v_i)$ □

STEP 1 IMPLIES :

(24)

IF

$G' < G$ AND $\chi(G')$ IS HUGE,

THEN

$\exists v \in V(G')$ S.T. $\chi(N^2(v))$ IS LARGE

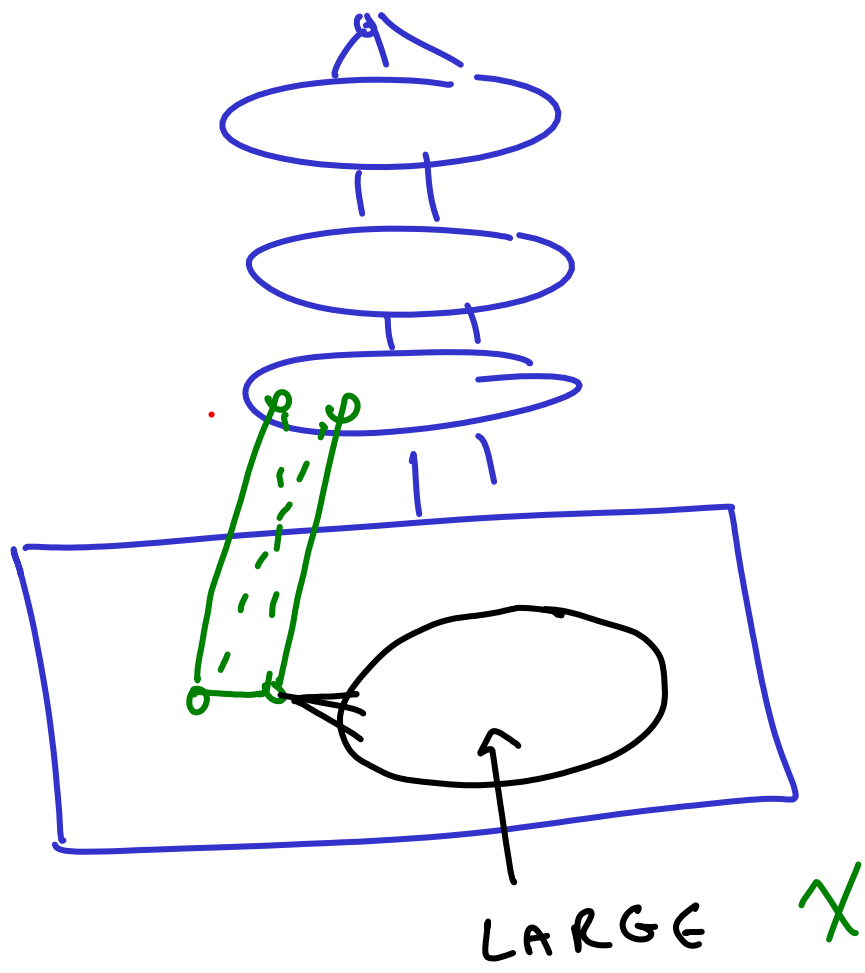
STEP 2 $\forall v \chi(N^2(v)) \leq K_{e,w}$

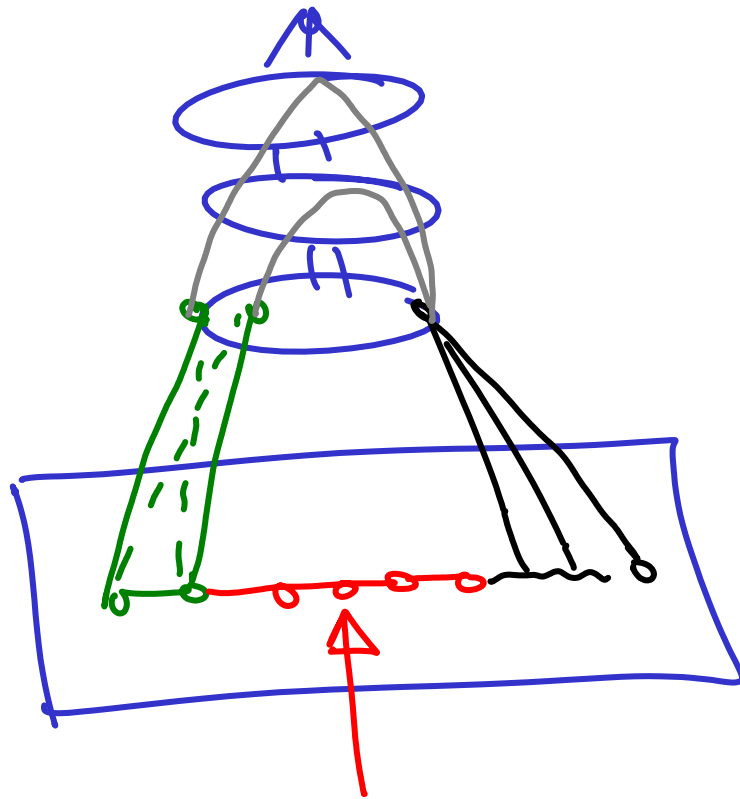


THM $\forall \ell$ THE FAMILY OF GRAPHS
WITH NO ODD HOLE OF LENGTH $\geq \ell$
IS χ -BOUNDED

PROOF INDUCTION ON ω

STEP 1 ASSUME $\forall \omega \chi(N^2\omega) \leq K$





GYÁRFÁS PÁTHS

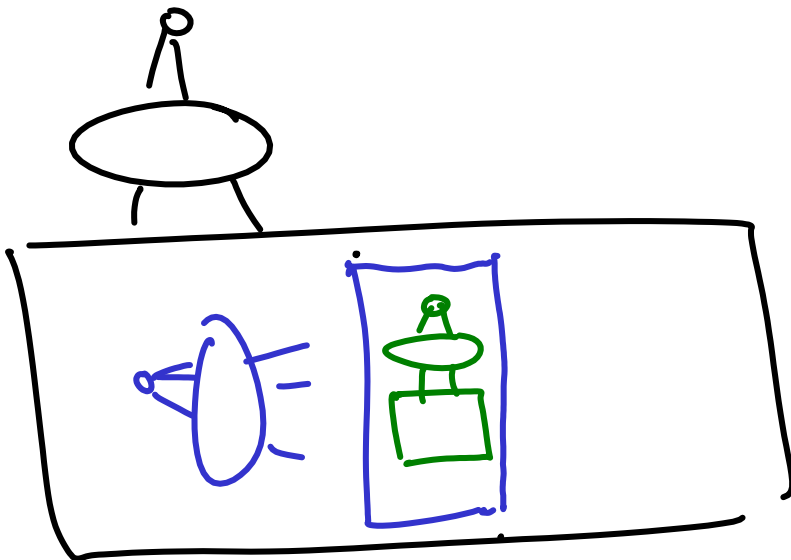
STEP 2

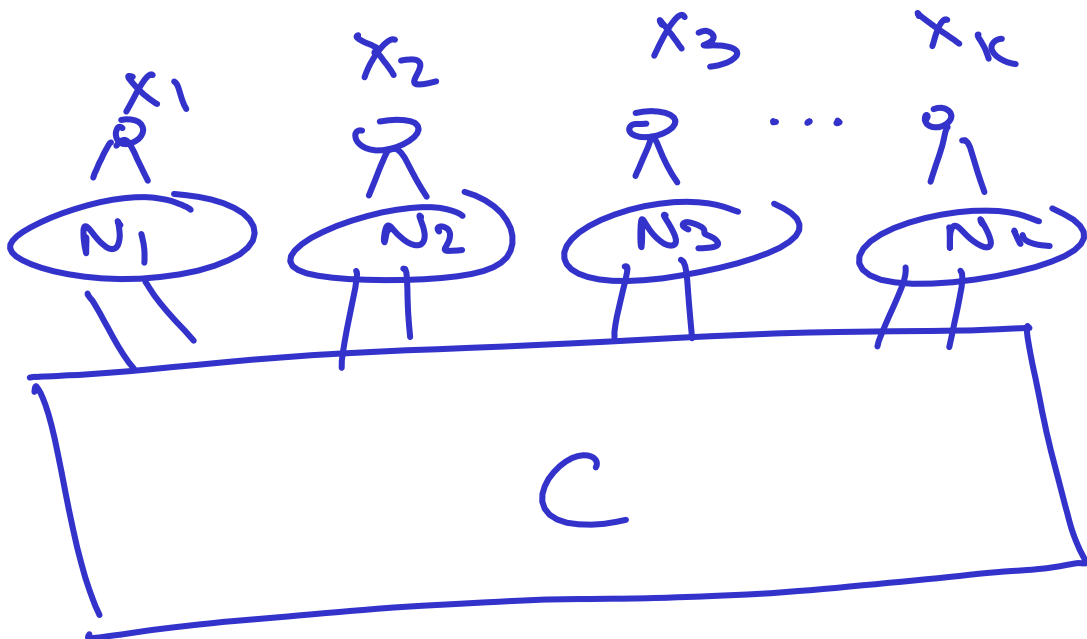
(27)

KNOW: $\forall G' < G$ WITH $\chi(G')$ HUGE
 $\Rightarrow v \in V(G')$ WITH $\chi(N^2(v))$ LARGE

PROVE: \Rightarrow ODD HOLE OF LENGTH $\geq p$

ITERATED COVERS:

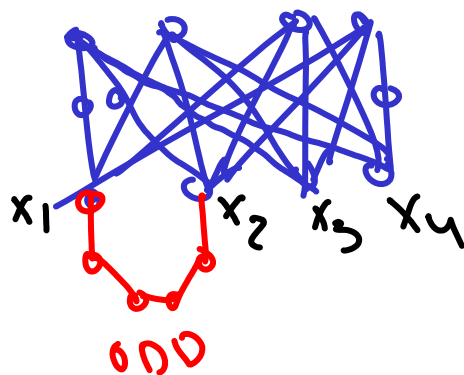




$X(C)$ BIG

$\forall c \in C \ \forall i \in \{1, \dots, k\} \ c \text{ HAS A NBR IN } N_i$

GET



OR LONG ODD HOLE IN $\cup_i N_i \cup \{x_i\}$